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Pseudospectral Solutions of Laminar Heat Transfer Problems in Pipelines

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Problems of laminar heat transfer associated with pipeline transport of oil under laminar flow conditions are solved by a relatively new numerical technique. This technique accommodates for the peculiarity of the domain of solution (one dimension, the axial, even when it is non-dimensionalized, is far more extended than the other two) and exhibits remarkable features in terms of computational economy, stability and accuracy.

The main interest in this paper is the adaptation of the method to problems with different boundary conditions approximating real pipelines (offshore, insulated, buried).

INTRODUCTION

The problem of determining the power requirements in a pipeline operating under non-isothermal conditions is tied to the problem of determining the heat transfer from/to the pipeline. There are three important aspects in laminar heat transfer problems in pipelines:

1. Realistic thermal boundary conditions (offshore, insulated, buried, arctic pipelines);

2. Markedly varying with temperature physical properties of the oil (viscosity); and

3. Significant free convection, comparable to forced convection under special conditions (non-insulated offshore pipelines).

The above aspects individually or combined render the solution to the equations governing laminar heat transfer to be numerical [1].

In laminar flows, in the absence of significant mixing, steep changes in physical properties and field variables are realized over the pipeline cross section. Accurate determination of heat transfer and pressure in pipelines is subject to accurate determination of the steep gradients of the field variables in the wall region. To accommodate for this, finite difference and finite element schemes employ finer grids as the pipe wall is approached. The use of non-uniform grids in some methods is not free of problems [2].

Finally, the domain of solution for heat transfer problems in pipelines is peculiar.

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One of the dimensions (axial) is far more extended than the others. This presents a challenge to numerical schemes in terms of computational economy, stability and accuracy.

The pseudospectral method, developed by Orszag and his co-workers [3–5], is basically a method of numerically solving partial differential equations by use of the finite (Fast) Fourier Transform. This seems reasonable for problems with periodic boundary conditions but not for the problems discussed here, for which general type boundary conditions apply.

In this paper, we address the adaptation of the method to laminar heat transfer problems relevant to oil pipelines with different types of boundary conditions.

Heat Transfer Problems

Since the interest in this paper is the adaptation of the numerical method to different thermal boundary conditions, we chose to solve the problem of thermally developing laminar forced convection with

1. Constant wall temperature (classical Greatz problem; Dirchlet boundary condition);
2. Constant wall heat flux (Neumann boundary condition); and
3. A linear combination of the temperature and its derivative specified at the wall (mixed boundary condition).

The constant wall temperature approximates the thermal boundary condition for an offshore pipeline. The constant wall heat flux condition approximates, under certain conditions, the insulated pipe. Finally, the mixed condition describes a buried pipeline if the physical properties of the ground in which the pipeline is buried can be assumed approximately constant.

For these three problems the governing equation is

$$u_z \frac{\partial T}{\partial z} = \kappa \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial T}{\partial r} \right) \tag{1}$$

where κ is the thermal diffusivity and u_z follows a Poiseuille distribution. The initial condition for all three problems is

$$z = 0, \qquad T = T_0. \tag{2}$$

The boundary condition at the centerline is that of symmetry

$$r = 0, \qquad \partial T / \partial r = 0. \tag{3}$$

At the pipe wall, we have either

$$r = R, \qquad T = T_w \tag{4a}$$

or

$$r = R, \qquad -k\, \partial T/\partial r = q_w \tag{4b}$$

or

$$r = R, \qquad -k\, \partial T/\partial r = h(T - T_s) \tag{4c}$$

(k, T_w, T_s, q_w and h are constants; k is the thermal conductivity of the fluid, T_w the temperature of the pipe wall, T_s the ground temperature and q_w is the wall heat transfer rate). For all three problems, the derivatives in the radial direction appearing in the right-hand side of Eq. (1) are calulated from finite expansions of the temperature in terms of Chebyshev polynomials with the temperature calculated at the extrema of the highest (Nth) order polynomial. With this selection of grid points, the Chebyshev polynomial expansions reduce to Fourier cosine expansions. The fact that Chebyshev polynomial expansions are appropriate for the three problems can be shown as follows. The boundary conditions for these problems can be represented as:

$$x = -1, \qquad \alpha_1 F + \alpha_2\, dF/dx = 0 \tag{5a}$$

$$x = 1, \qquad \alpha_3 F + \alpha_4\, dF/dx = 0. \tag{5b}$$

If we consider a finite expansion of F of the form

$$F(x) = \sum_{n \geqslant 4}^{N} a_n q_n(x) \tag{6}$$

where

$$q_{2n+1} = T_{2n+1} - 2T_{2n-1} + T_{2n-3} - T_3 + T_1, \qquad n \geqslant 2 \tag{7a}$$

$$q_{2n} = T_{2n} - 2T_{2n-2} + T_{2n-4} - 2T_2 + 2T_0, \qquad n \geqslant 2, \tag{7b}$$

since

$$q_{2n}(\pm 1) = q_{2n+1}(\pm 1) = q'_{2n}(\pm 1) = q'_{2n+1}(\pm 1) = 0, \tag{8}$$

the boundary conditions (5a) and (5b) are automatically satisfied. Furthermore, the expansion of Eq. (6) reduces to an expansion of the form

$$F(x) = \sum_{n=0}^{N} b_n T_n(x) \tag{9}$$

with

$$b_0 = a_4 + 2 \sum_{n=2}^{N/2} a_{2n} \tag{10a}$$

$$b_1 = a_5 + \sum_{n=2}^{N/2-1} a_{2n+1} \tag{10b}$$

$$b_2 = a_6 - 2a_4 - 2 \sum_{n=2}^{N/2} a_{2n} \tag{10c}$$

$$b_3 = -2a_5 + a_7 - \sum_{n=2}^{N/2-1} a_{2n+1} \tag{10d}$$

and

$$b_n = a_n - 2a_{n+2} + a_{n+4}, \qquad n \geqslant 4. \tag{10e}$$

The derivatives of F can be calculated by differentiating the expansion (9) term by term as

$$(\partial F/\partial x)_{x=x_n} = \sum_{k=0}^{N} b_k^{(1)} T_k(x_n) \tag{11a}$$

$$(\partial^2 F/\partial x^2)_{x=x_n} = \sum_{k=0}^{N} b_k^{(2)} T_k(x_n) \tag{11b}$$

with $b_k^{(1)}$ and $b_k^{(2)}$ determined from the recursive relation

$$c_{k-1} b_{k-1}^{(q)} - b_{k+1}^{(q)} = 2k b_k^{(q-1)}, \qquad k \geqslant 1 \tag{12}$$

with $c_0 = 2$ and $c_k = 1$ for $k > 0$.

From the Chebyshev polynomial expansions for the temperature the right-hand side of Eq. (1) is computed. A second-order predictor-corrector (Adams–Moulton) scheme is used to advance the solution in the axial direction from z to $z + \Delta z$ (semi-discrete approximation).

The solution to every problem is preceded by proper non-dimensionalization of the working variables in the form

$$r' = r/R; \qquad z' = z/(RRePr); \qquad u_z' = u_z/\langle U_z \rangle \tag{13}$$

(R is the pipe radius, $\langle U_z \rangle$ is the area-averaged velocity, $Re = 2R\langle U_z \rangle \rho/\mu$ is the Reynolds number, $Pr = \nu/\kappa$ is the Prandtl number), and

$$T' = \frac{T - T_w}{T_0 - T_w} \tag{14a}$$

(T_0 is the fluid temperature at the entrance) for the constant wall temperature problem,

$$T' = \frac{T - T_0}{Rq_w/k} \tag{14b}$$

for the constant wall heat flux problem and

$$T' = \frac{T - (T_s + \alpha\sqrt{H^2 - R^2})}{T_0 - (T_s + \alpha\sqrt{H^2 - R^2})} \tag{14c}$$

(α is the geothermal gradient and H is the burial depth) for the buried pipe problem.

With this non-dimensionalization, Eq. (1) takes the form

$$(1 - r'^2)\frac{\partial T'}{\partial z'} = \frac{1}{r'}\frac{\partial}{\partial r'}\left(r'\frac{\partial T'}{\partial r'}\right). \tag{15}$$

Although the right-hand side of Eq. (1) for all these problems is evaluated from the same type of expansion, the computational procedure is different for each problem. Each problem, then, will be dealt with separately below.

Constant Wall Temperature

The boundary conditions for the constant wall temperature problem are

$$r' = 0, \qquad \frac{\partial T'}{\partial r'} = 0 \tag{16a}$$

and

$$r' = 1, \qquad T' = 0. \tag{16b}$$

For this problem, we define a new variable r'' as

$$\begin{aligned} r'' &= r' \qquad \text{for} \quad r'' \in [0, 1] \\ r'' &= -r' \qquad \text{for} \quad r'' \in [-1, 0). \end{aligned} \tag{17}$$

With this definition, the symmetry boundary condition becomes

$$T'(-r'') = T'(r''), \qquad r'' \in (0, 1] \tag{18a}$$

and the boundary condition (16b) reduces to

$$r'' = \pm 1, \qquad T' = 0. \tag{18b}$$

Expansion of T' in terms of Chebyshev polynomials,

$$T'(r_i, z) = \sum_{n=0}^{N} a_n(z)\, T_n(r_i) \tag{19}$$

with $r_i = \cos \pi i/N$, reduces to a Fourier cosine series expansion which automatically satisfies both boundary conditions (18a) and (18b).

At $r' = 0$, Eq. (15) has a removable singularity. The same equation has a non-removable second-order singularity at $r' = 1$. However, Eq. (15) need not be used for calculation of T' at $r' = 1$, since T' is specified there by the boundary condition. This is not the case for the constant wall heat flux problem.

CONSTANT WALL HEAT FLUX

For this problem, a special procedure is devised to overcome the singularity problem in the evaluation of temperature at the wall. Differentiating with respect to r' Eq. (15) together with the initial boundary conditions for the constant wall heat flux problem, we obtain

$$(1-r'^2)^2 \frac{\partial q'}{\partial z'} = \left(3 - \frac{1}{r'^2}\right) q' + \left(r' + \frac{1}{r'}\right) \frac{\partial q'}{\partial r'} + (1-r'^2) \frac{\partial^2 q'}{\partial r'^2} \tag{20}$$

$$z' = 0, \qquad q' = 0 \tag{21a}$$

$$z' = 0, \qquad r' = 0, \qquad q' = 0 \tag{21b}$$

$$r' = 1, \qquad q' = 1 \tag{21c}$$

where

$$q' = -\frac{\partial T'}{\partial r'}.$$

In accordance with the previously outlined procedure the derivatives of q' in the right-hand side of Eq. (20) are evaluated from Chebyshev polynomial expansions of q'. The singularity problem has been alleviated since Eq. (20) need not be used at $r' = 1$ to calculate q' there. However, the solution of Eq. (20) together with initial and boundary conditions (21a)–(21c) yields $\partial T'/\partial r'$ and not T'. An inverse problem develops as now, knowing the coefficients in

$$\partial T'/\partial r' = \sum_{n=0}^{N} a_n^{(1)} T_n(r'), \tag{22}$$

we would like to determine the expansion coefficients in

$$T' = \sum_{n=0}^{N} a_n T_n(r'). \tag{23}$$

The coefficients a_n are related to $a_n^{(1)}$ coefficients through Eq. (12). The latter permits evaluation of a_n for $n = 1, 2, \ldots, N$. We still need a_0 to complete the expansion (23), and this is obtained in the following way.

The dimensionless bulk temperature at any cross section is defined as

$$T'_b = \frac{\int_0^1 T' u'_z r' \, dr'}{\int_0^1 u'_z r' \, dr'}. \tag{24}$$

The bulk temperature for the problem of interest reduces to

$$T'_b = 4 \int_0^1 \hat{T}'(1-r'^2) \, r' \, dr' + a_0 \tag{25}$$

with $\hat{T}'$ given by

$$\hat{T}' = \sum_{n=1}^{N} a_n T_n(r'). \tag{26}$$

(All the coefficients a_n in Eq. (26) can be found.) The integrand of Eq. (25) can be expressed in terms of Chebyshev polynomials. If the integrand is represented as

$$F(r') = \sum_{n=0}^{N} b_n T_n(r') \tag{27}$$

making use of the properties of the Chebyshev polynomial [6] we have

$$\int_0^{r'} F(r')\,dr' = \sum_{n=0}^{N+1} c_n T_n(r') \tag{28}$$

with

$$c_{N+1} = \frac{b_N}{2(N+1)}; \qquad c_N = \frac{b_{N-1}}{2N} \tag{29a}$$

$$c_n = \frac{1}{2n}(b_{n-1} - b_{n+1}), \qquad n = 1,\ldots, N-1 \tag{29b}$$

and

$$c_0 = 2[c_1 - c_2 + \cdots + (-1)^N c_{N+1}]. \tag{29c}$$

A macroscopic energy balance for the constant wall heat flux problem yields another relation for T_b', namely,

$$T_b' = 4z'. \tag{30}$$

From Eqs. (25) and (30), a_0 can be determined. Hence, T' is completely determined from the expansion (23) throughout the domain of solution.

Buried Pipe

The boundary condition at the wall for this problem reduces to

$$r' = 1, \qquad \partial T'/\partial r' = -BT' \tag{31}$$

with

$$B = \frac{k_s}{k \ln[H/R + \sqrt{(H/R)^2 - 1}]}$$

(B is the Biot number and k_s the thermal conductivity of the soil). This boundary condition is provided by the solution of the external buried pipe heat transfer problem [7]. Again, T' is expanded in terms of Chebyshev polynomials and is chosen to be evaluated at the extrema of the Nth-order polynomial. The coefficients of the expansion are calculated as follows. At each time step, for the interior points $i = i, \ldots, N-1$ (total number of points $N+1$), we have

$$\sum_{n=0}^{N} a_n T_n(r_i') = T(r_i'), \qquad i = 1, \ldots, N-1. \tag{32}$$

The symmetry boundary condition yields

$$\sum_{n=0}^{N} a_n \frac{dT_n(0)}{dr'} = \sum_{k=0}^{N/2} a_{2k+1}(-1)^k (2k+1) = 0 \tag{33}$$

$$\left(\frac{dT_{2k}(0)}{dr'} = 0 \text{ for any } k\right).$$

Finally, the wall boundary condition (31) can be written as

$$\sum_{n=0}^{N} a_n \frac{dT_n(1)}{dr'} = -B \sum_{n=0}^{N} a_n T_n(1). \tag{34}$$

Equation (34), making use of the properties of the Chebyshev polynomials [6], reduces to

$$\sum_{n=0}^{N} a_n(n^2 - B) = 0 \tag{35}$$

$$\left(\frac{d^p T_k(\pm 1)}{dx^p} = (\pm 1)^{k+p} \prod_{m=0}^{p-1} (k^2 - m^2)/(2m+1)\right).$$

Equations (32), (33) and (34) constitute a system of $N+1$ equations for the coefficients a_n ($n = 0, \ldots, N$). Consequently, the temperature field is evaluated in the following way. At each time step we solve the energy equation in the interior of the radial domain (0, 1). From the system of Eqs. (32), (33) and (35) the coefficients in the expansion are determined. Finally, the temperatures at the centerline and the wall are obtained from the expansion (19) for $r_i' = 0$ and $r_i' = 1$, respectively.

Results

From the temperature field the local Nusselt numbers corresponding to these three problems were generated according to

$$Nu = -2/T_b'(\partial T'/\partial r')_{r'=1} \tag{36a}$$

$$Nu = 2/(T_w' - 4z') \tag{36b}$$

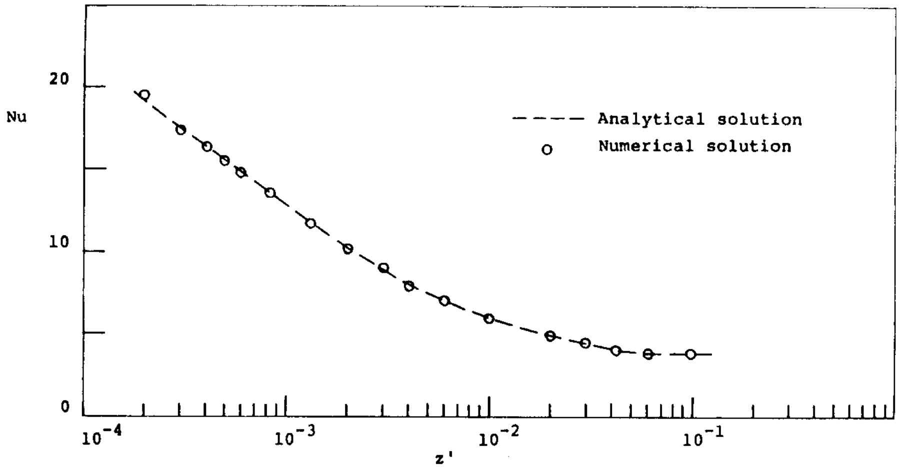

FIG. 1. Nusselt number as a function of position. Constant wall temperature.

and

$$Nu = 2/(T'_w - T'_b)(\partial T'/\partial r')_{r'=1} \tag{36c}$$

respectively, and T'_b defined by Eq. (24).

The local Nusselt numbers for the constant wall temperature problem (offshore pipeline) are shown in Fig. 1. Complete agreement with the values generated from the analytical solution [8] is found. For $z' \simeq 0.10$ the local Nusselt number approaches its asymptotic value 3.66 [9].

In Fig. 2, comparison of the local Nusselt numbers from the pseudospectral

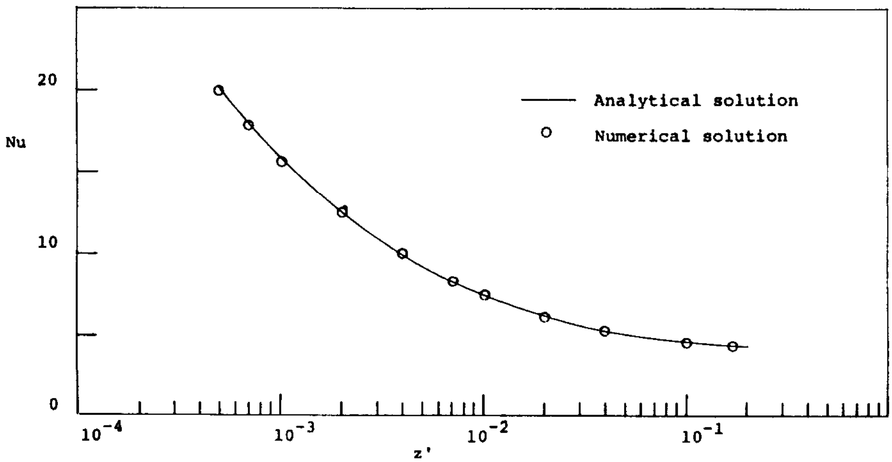

FIG. 2. Nusselt number as a function of position. Constant wall heat flux.

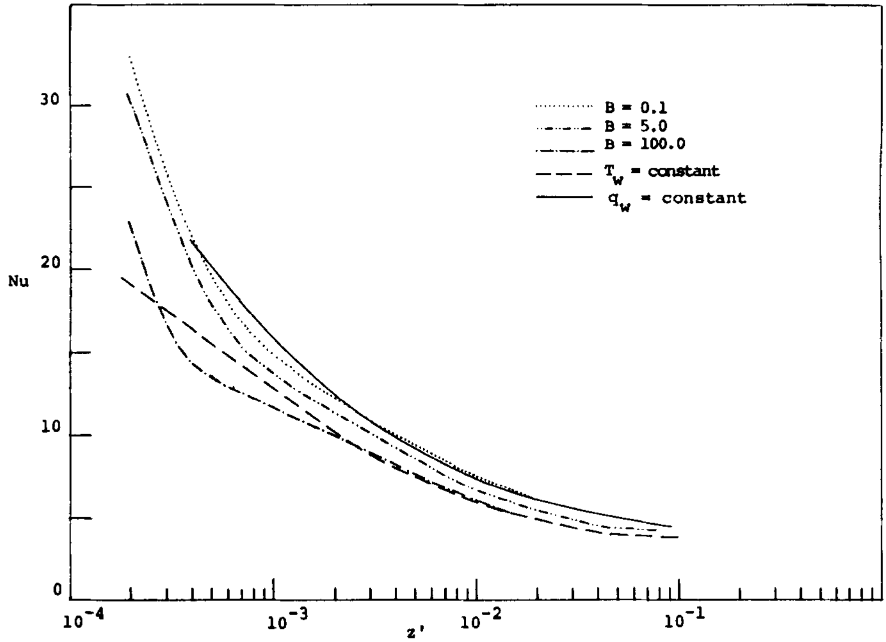

FIG. 3. Nusselt number as a function of position. Buried pipe.

solution with their values from the analytical solution [10] is shown. Again, the agreement is remarkable.

The local Nusselt numbers for the buried pipe, for different values of the Biot number B, are shown in Fig. 3. As expected, for small B the local Nusselt number approaches asymptotically the curve for the constant wall heat flux problem, while for large B the local Nusselt number curve approaches asymptotically the curve for the constant wall temperature problem.

The calculations were done on a UNIVAC 1100 machine. We checked for stability by repeating part of the calculations for different axial step sizes and number of points in the radial direction. This mesh refinement check shows our scheme to be numerically stable and convergent. The distribution of the points employed by the pseudospectral method becomes denser as the pipe wall is approached. However, this does not affect the stability of the method as in finite difference schemes employing non-uniform grids [2].

The calculations were carried out with fine points in the radial direction. The results are obtained with accuracy comparable to that of finite difference calculations with approximately 20 points in the radial direction. According to Gottlieb and Orszag [5], pseudospectral methods employing N-term expansions achieve accuracy comparable to finite difference schemes employing N^2 grid points.

Conclusions

Laminar frced convection problems associated with long pipelines were solved by the pseudospectral method. The main interest in this paper was the adaptation of the method for three common types of thermal boundary conditions (Dirichlet, Neumann and mixed) which correspond to different types of pipeline environment (offshore, insulated, buried).

The results of the pseudospectral solution compare well with the results of previously reported solutions to the problems examined here.

The method exhibits computational economy features particularly desired in calculations of pipelines in which the axial direction is far more extended than the redial one. The effective distribution of collocation points (denser distribution in the vicinity of the wall) is the key to explaining the accuracy of the method in determining heat transfer rates at the wall. Although a non-uniform grid is employed, the pseudospectral method does not suffer the stability problems of other methods employing non-uniform grids.

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